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LIMIT THEOREMS FOR ONE AND TWO-DIMENSIONAL RANDOM WALKS IN RANDOM SCENERY

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Abstract. Random walks in random scenery are processes defined by $Z_n := \sum_{k=1}^n \xi_{X_1+\dots+X_k}$, where $(X_k, k \geq 1)$ and $(\xi_y, y \in \mathbb{Z}^d)$ are two independent sequences of i.i.d. random variables with values in $\mathbb{Z}^d$ and $\mathbb{R}$ respectively. We suppose that the distributions of X_1 and ξ_0 belong to the normal basin of attraction of stable distribution of index $\alpha \in (0, 2]$ and $\beta \in (0, 2]$. When $d = 1$ and $\alpha \neq 1$, a functional limit theorem has been established in [11] and a local limit theorem in [5]. In this paper, we establish the convergence of the finite-dimensional distributions and a local limit theorem when $\alpha = d$ (i.e. $\alpha = d = 1$ or $\alpha = d = 2$) and $\beta \in (0, 2]$. Let us mention that functional limit theorems have been established in [2] and recently in [8] in the particular case where $\beta = 2$ (respectively for $\alpha = d = 2$ and $\alpha = d = 1$).

1. Introduction

Random walks in random scenery (RWRS) are simple models of processes in disordered media with long-range correlations. They have been used in a wide variety of models in physics to study anomalous dispersion in layered random flows [14], diffusion with random sources, or spin depolarization in random fields (we refer the reader to Le Doussal's review paper [12] for a discussion of these models).

On the mathematical side, motivated by the construction of new self-similar processes with stationary increments, Kesten and Spitzer [11] and Borodin [3, 4] introduced RWRS in dimension one and proved functional limit theorems. This study has been completed in many works, in particular in [2] and [8]. These processes are defined as follows. Let $\xi := (\xi_y, y \in \mathbb{Z}^d)$ and $X := (X_k, k \geq 1)$ be two independent sequences of independent identically distributed random variables taking values in $\mathbb{R}$ and $\mathbb{Z}^d$ respectively. The sequence ξ is called the *random scenery*. The sequence X is the sequence of increments of the *random walk* $(S_n, n \geq 0)$ defined by $S_0 := 0$ and $S_n := \sum_{i=1}^n X_i$, for $n \geq 1$. The *random walk in random scenery* Z is then defined by

$$Z_0 := 0 \text{ and } \forall n \geq 1,\ Z_n := \sum_{k=0}^{n-1} \xi_{S_k}.$$

Denoting by $N_n(y)$ the local time of the random walk S :

$$N_n(y) := \#\{k = 0, ..., n-1 \ : \ S_k = y\},$$

it is straightforward to see that Z_n can be rewritten as $Z_n = \sum_y \xi_y N_n(y)$.

As in [11], the distribution of ξ_0 is assumed to belong to the normal domain of attraction of a strictly stable distribution $\mathcal{S}_\beta$ of index $\beta \in (0, 2]$, with characteristic function ϕ given by

$$\phi(u) = e^{-|u|^\beta (A_1 + iA_2 \operatorname{sgn}(u))} \quad u \in \mathbb{R},$$

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where $0 < A_1 < \infty$ and $|A_1^{-1}A_2| \le |\tan(\pi\beta/2)|$. We will denote by φ_ξ the characteristic function of the ξ_x's. When $\beta > 1$, this implies that $\mathbb{E}[\xi_0] = 0$. When $\beta = 1$, we will further assume the symmetry condition

$$\sup_{t>0} \left|\mathbb{E}\left[\xi_0 \ \mathbb{1}_{\{|\xi_0|\le t\}}\right]\right| < +\infty\,. \tag{1}$$

Under these conditions (for $\beta \in (0;2]$), there exists $C_\xi > 0$ such that we have

$$\forall t > 0, \quad \mathbb{P}\left(|\xi_0| \ge t\right) \le C_\xi t^{-\beta}. \tag{2}$$

Concerning the random walk, the distribution of X_1 is assumed to belong to the normal basin of attraction of a stable distribution $\mathcal{S}'_\alpha$ with index $\alpha \in (0,2]$.

Then the following weak convergences hold in the space of càdlàg real-valued functions defined on $[0,\infty)$ and on $\mathbb{R}$ respectively, endowed with the Skorohod J_1-topology (see [1, chapter 3]) :

$$\left(n^{-1/\alpha} S_{\lfloor nt\rfloor}\right)_{t\ge 0} \overset{\mathcal{L}}{\underset{n\to\infty}{\Longrightarrow}} (U(t))_{t\ge 0}$$

$$\text{and} \qquad \left(n^{-\frac{1}{\beta}} \sum_{k=0}^{\lfloor nx\rfloor} \xi_{ke_1}\right)_{x\in\mathbb{R}} \overset{\mathcal{L}}{\underset{n\to\infty}{\Longrightarrow}} (Y(x))_{x\in\mathbb{R}}\,, \text{ with } e_1 = (1,0,\cdots,0) \in \mathbb{Z}^d\,,$$

where U and Y are two independent Lévy processes such that $U(0) = 0$, $Y(0) = 0$, $U(1)$ has distribution $\mathcal{S}'_\alpha$, $Y(1)$ and $Y(-1)$ have distribution $\mathcal{S}_\beta$.

Functional limit theorem.
Our first result is concerned with a functional limit theorem for $(Z_{[nt]})_{t\ge 0}$. Intuitively speaking,

- when $\alpha < d$, the random walk S_n is transient, its range is of order n, and Z_n has the same behaviour as a sum of about n independent random variables with the same distribution as the variables ξ_x. Therefore, $n^{-1/\beta}(Z_{[nt]})_{t\ge 0}$ weakly converges in the space $D([0,\infty))$ of càdlàg functions endowed with the Skorohod J_1-topology, to a multiple of the process (Y_t), as proved in [4];
- when $\alpha > d$ (i.e $d = 1$ and $1 < \alpha \le 2$), the random walk S_n is recurrent, its range is of order $n^{1/\alpha}$, its local times are of order $n^{1-1/\alpha}$, so that Z_n is of order $n^{1-\frac{1}{\alpha}+\frac{1}{\alpha\beta}}$. In this situation, [3] and [11] proved a functional limit theorem for $n^{-(1-\frac{1}{\alpha}+\frac{1}{\alpha\beta})}(Z_{[nt]})_{t\ge 0}$ in the space $\mathbb{C}([0,\infty))$ of continuous functions endowed with the uniform topology , the limiting process being a self-similar process, but not a stable one.
- when $\alpha = d$ (i.e. $\alpha = d = 1$, or $\alpha = d = 2$), S_n is recurrent, its range is of order $n/\log(n)$, its local times are of order $\log(n)$ so that Z_n is of order $n^{\frac{1}{\beta}}\log(n)^{\frac{\beta-1}{\beta}}$. In this situation, a functional limit theorem in the space of continuous functions was proved in [2] for $d = \alpha = \beta = 2$, and in [8] for $d = \alpha = 1$ and $\beta = 2$.

Our first result gives a limit theorem for $\alpha = d$ (and so $d \in \{1,2\}$) and for any value of $\beta \in (0;2)$ in the finite distributional sense.

Theorem 1. *Let us assume that $\beta \in (0;2]$ and that*

(a) *either $d = 2$ and X_1 is centered, square integrable with invertible variance matrix Σ and then we define $A := 2\sqrt{\det\Sigma}$;*
(b) *or $d = 1$ and $\left(\frac{S_n}{n}\right)_n$ converges in distribution to a random variable with characteristic function given by $t \mapsto \exp(-a|t|)$ with $a > 0$ and then we define $A := a$.*

Then, the finite-dimensional distributions of the sequence of random variables

$$\left(\left(\frac{Z_{[nt]}}{n^{1/\beta}\log(n)^{(\beta-1)/\beta}}\right)_{t\geq 0}\right)_{n\geq 2}$$

converges to the finite-dimensional distributions of the process

$$\left(\tilde{Y}_t := \left(\frac{\Gamma(\beta+1)}{(\pi A)^{\beta-1}}\right)^{1/\beta} Y(t)\right)_{t\geq 0}.$$

Moreover, if $\beta < 2$, the sequence

$$\left(\left(\frac{Z_{[nt]}}{n^{1/\beta}\log(n)^{(\beta-1)/\beta}}\right)_{t\geq 0}\right)_{n\geq 2}$$

is not tight in $\mathcal{D}([0,\infty))$ endowed with the J_1-topology.

Local limit theorem.

Our next results concern a local limit theorem for $(Z_n)_n$. The $d=1$ case was treated in [5] for $\alpha \in (0;2]\setminus\{1\}$ and all values of $\beta \in (0;2]$. Here, we complete this study by proving a local limit theorem for $\alpha = d = 1$ (and $\beta \in (0;2]$). By a direct adaptation of the proof of this result, we also establish a local limit theorem for $\alpha = d = 2$ (we just adapt the definition of "peaks", see section 3.5). Let us notice that the same adaptation can be done from [5] (case $\alpha < 1$) to get local limit theorems for $d \geq 2$, $\alpha < d$ and $\beta \in (0;2]$.

We give two results corresponding respectively to the case when ξ_0 is lattice and to the case when it is strongly non-lattice. We denote by φ_ξ the characteristic function of ξ_0.

Theorem 2. *Assume that ξ_0 takes its values in $\mathbb{Z}$. Let $d_0 \geq 1$ be the integer such that $\{u : |\varphi_\xi(u)| = 1\} = \frac{2\pi}{d_0}\mathbb{Z}$. Let $b_n := n^{1/\beta}(\log(n))^{(\beta-1)/\beta}$. Under the previous assumptions on the random walk and on the scenery, for $\alpha = d \in \{1,2\}$, for every $\beta \in (0,2]$, and for every $x \in \mathbb{R}$,*

- *if $\mathbb{P}(n\xi_0 - \lfloor b_n x\rfloor \notin d_0\mathbb{Z}) = 1$, then $\mathbb{P}(Z_n = \lfloor b_n x\rfloor) = 0$;*
- *if $\mathbb{P}(n\xi_0 - \lfloor b_n x\rfloor \in d_0\mathbb{Z}) = 1$, then*

$$\mathbb{P}(Z_n = \lfloor b_n x\rfloor) = d_0\frac{C(x)}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} + o(n^{-1/\beta}(\log(n))^{-(\beta-1)/\beta})$$

uniformly in $x \in \mathbb{R}$, where $C(\cdot)$ is the density function of $\tilde{Y}_1$.

Theorem 3. *Assume now that ξ_0 is strongly non-lattice which means that*

$$\limsup_{|u|\to+\infty} |\varphi_\xi(u)| < 1.$$

We still assume that $\alpha = d \in \{1,2\}$ and $\beta \in (0;2]$. Then, for every $x, a, b \in \mathbb{R}$ such that $a < b$, we have

$$\lim_{n\to+\infty} b_n \mathbb{P}(Z_n \in [b_n x + a; b_n x + b]) = C(x)(b-a),$$

with $b_n := n^{1/\beta}(\log(n))^{(\beta-1)/\beta}$ and where $C(\cdot)$ is the density function of $\tilde{Y}_1$.

2. Proof of the limit theorem

Before proving the theorem, we prove some technical lemmas. For any real number $\gamma > 0$, any integer $m \geq 1$, any $\theta_1, \dots, \theta_m \in \mathbb{R}$, any $t_0 = 0 < t_1 < \dots < t_m$, we consider the sequences of random variables $(L_n(\gamma))_{n\geq 2}$ and $(L'_n(\gamma))_{n\geq 2}$ defined by

$$L_n(\gamma) := \frac{1}{n(\log n)^{\gamma-1}} \sum_{x\in\mathbb{Z}^d} \left| \sum_{i=1}^m \theta_i (N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)) \right|^\gamma$$

and

$$L'_n(\gamma) := \frac{1}{n(\log n)^{\gamma-1}} \sum_{x\in\mathbb{Z}^d} \left| \sum_{i=1}^m \theta_i (N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)) \right|^\gamma \operatorname{sgn} \left(\sum_{i=1}^m \theta_i (N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)) \right).$$

Lemma 4. *For any real number $\gamma > 0$, any integer $m \geq 1$, any $\theta_1, \dots, \theta_m \in \mathbb{R}$, any $t_0 = 0 < t_1 < \dots < t_m$, the following convergences hold $\mathbb{P}$-almost surely*

$$\lim_{n\to+\infty} L_n(\gamma) = \frac{\Gamma(\gamma+1)}{(\pi A)^{\gamma-1}} \sum_{i=1}^m |\theta_i|^\gamma (t_i - t_{i-1}) \tag{3}$$

and

$$\lim_{n\to+\infty} L'_n(\gamma) = \frac{\Gamma(\gamma+1)}{(\pi A)^{\gamma-1}} \sum_{i=1}^m |\theta_i|^\gamma sgn(\theta_i) (t_i - t_{i-1}). \tag{4}$$

Proof. We fix an integer $m \geq 1$ and $2m$ real numbers $\theta_1, \dots, \theta_m, t_1, ..., t_m$ such that $0 < t_1 < \dots < t_m$ and we set $t_0 := 0$. To simplify notations, we write $b_{i,n}(x) := N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)$. Following the techniques developed in [6], we first have to prove (3) and (4) for integer γ: for every integer $k \geq 1$, $\mathbb{P}$-almost surely, as n goes to infinity, we have

$$\frac{1}{n(\log n)^{k-1}} \sum_{x\in\mathbb{Z}^d} \left(\sum_{i=1}^m \theta_i b_{i,n}(x) \right)^k \longrightarrow \frac{\Gamma(k+1)}{(\pi A)^{k-1}} \sum_{i=1}^m \theta_i^k (t_i - t_{i-1}). \tag{5}$$

Let us assume (5) for a while, and let us end the proof of (3) and (4) for any positive real γ. Given the random walk $S := (S_n)_n$, let $(U_n)_{n\geq 1}$ be a sequence of random variables with values in $\mathbb{Z}^d$, such that for all n, U_n is a point chosen uniformly in the range of the random walk up to time $[nt_m]$, that is

$$\mathbb{P}(U_n = x \big| S) = R_{[nt_m]}^{-1} \mathbf{1}_{\{N_{[nt_m]}(x)\geq 1\}},$$

with $R_k := \#\{y : N_k(y) > 0\}$. Moreover, let U' be a random variable with values in $\{1, \dots, m\}$ and distribution

$$\mathbb{P}(U' = i) = (t_i - t_{i-1})/t_m$$

and let T be a random variable with exponential distribution with parameter one and independent of U'.

Then, for $\mathbb{P}-$ almost every realization of the random walk S, the sequence of random variables

$$\left(W_n := \frac{\pi A}{\log(n)} \sum_{i=1}^m \theta_i b_{i,n}(U_n) \right)_n$$

converges in distribution to the random variable $W := \theta_{U'} T$. Indeed, the moment of order k of W_n given S is

$$\mathbb{E}(W_n^k | S) \;=\; \frac{(\pi A)^k}{n(\log n)^{k-1}} \sum_{x\in\mathbb{Z}^d} \left(\sum_{i=1}^m \theta_i b_{i,n}(x) \right)^k \frac{n}{\log(n) R([nt_m])}.$$

Using (5) and the fact that $((\log n)R_n/n)_n$ converges almost surely to πA (see [9, 13]), the moments $\mathbb{E}(W_n^k|S)$ converges a.s. to $\mathbb{E}(W^k) = \Gamma(k+1)\sum_{i=1}^m \theta_i^k(t_i-t_{i-1})/t_m$, which proves the convergence in distribution of $(W_n)_n$ (given S) to W. This ensure, in particular, the convergence in distribution of $(|W_n|^\gamma)_n$ and of $(|W_n|^\gamma \mathrm{sgn}(W_n))_n$ (given S) to $|W|^\gamma$ and $|W|^\gamma\mathrm{sgn}(W)$ respectively (for every real number $\gamma \geq 0$ and for $\mathbb{P}-$ almost every realization of the random walk S). Since any moment of $|W_n|$ can be bounded from above by an integer moment, we deduce that, for any $\gamma \geq 0$, we have $\mathbb{P}$-almost surely

$$\lim_{n\to+\infty} \mathbb{E}(|W_n|^\gamma|S) = \mathbb{E}(|W|^\gamma) \quad \text{and} \quad \lim_{n\to+\infty} \mathbb{E}(|W_n|^\gamma \mathrm{sgn}(W_n)|S) = \mathbb{E}(|W|^\gamma\mathrm{sgn}(W)),$$

which proves lemma 4.
Let us prove (5). Let $k \geq 1$. According to Theorem 1 in [6] (proved for $\alpha = d = 2$, but also valid for $\alpha = d = 1$), we have

$$\forall i \in \{1,...,m\}, \quad \lim_{n\to+\infty} \frac{1}{n(\log n)^{k-1}} \sum_{x\in\mathbb{Z}^d} (b_{i,n}(x))^k = \frac{\Gamma(k+1)}{(\pi A)^{k-1}}(t_i - t_{i-1}), \ \mathbb{P}-a.s.. \tag{6}$$

We define

$$\Sigma_n(\theta_1,...,\theta_m) := \sum_{x\in\mathbb{Z}^d}\left(\sum_{i=1}^m \theta_i b_{i,n}(x)\right)^k - \sum_{x\in\mathbb{Z}^d}\sum_{i=1}^m (\theta_i)^k\,(b_{i,n}(x))^k\,. \tag{7}$$

According to (6), it is enough to prove that $\mathbb{P}-$a.s., $\Sigma_n(\theta_1,...,\theta_m) = o(n(\log n)^{k-1})$. We observe that $\Sigma_n(\theta_1,...,\theta_m)$ is the sum of the following terms

$$\sum_{x\in\mathbb{Z}^d}\prod_{j=1}^k \left(\theta_{i_j} b_{i_j,n}(x)\right). \tag{8}$$

over all the k-tuple $(i_1,\ldots,i_k) \in \{1,\ldots,m\}^k$, with at least two distinct indices. We observe that

$$|\Sigma_n(\theta_1,...,\theta_m)| \leq \max(|\theta_1|,...,|\theta_m|)^k \Sigma_n(1,...,1).$$

But, we have

$$\begin{aligned}
\Sigma_n(1,...,1) &= \sum_{x\in\mathbb{Z}^d}\left(N_{[nt_m]}(x)\right)^k - \sum_{x\in\mathbb{Z}^d}\sum_{i=1}^m (b_{i,n}(x))^k \\
&= \sum_{x\in\mathbb{Z}^d}\left(N_{[nt_m]}(x)\right)^k - \sum_{i=1}^m\sum_{x\in\mathbb{Z}^d} (b_{i,n}(x))^k = o(n\log(n)^{k-1}),
\end{aligned}$$

according to (6). □

Lemma 5. *For any $\rho > 0$,*

$$\sup_{x\in\mathbb{Z}^d} N_n(x) = o(n^\rho) \quad \text{a.s..}$$

Proof. See Lemma 2.5 in [2]. □

Proof of Theorem 1. Let an integer $m \geq 1$ and $2m$ real numbers $\theta_1,...,\theta_m, t_1,...,t_m$ such that $0 < t_1 < ... < t_m$. We set $t_0 := 0$. Again, we use the notation $b_{i,n}(x) := N_{[nt_i]}(x) - N_{[nt_{i-1}]}(x)$. Let us write $\tilde{Z}_n := \frac{1}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}}\sum_{i=1}^m \theta_i(Z_{[nt_i]} - Z_{[nt_{i-1}]})$. We have to prove that

$$\mathbb{E}[e^{i\tilde{Z}_n}] \to \prod_{i=1}^m \phi\left(\theta_i(t_i-t_{i-1})^{1/\beta}\left(\frac{\Gamma(\beta+1)}{(\pi A)^{\beta-1}}\right)^{1/\beta}\right), \tag{9}$$

as n goes to infinity. We observe that $\tilde{Z}_n = \frac{1}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} \sum_{x\in\mathbb{Z}^d}\sum_{i=1}^m \theta_i b_{i,n}(x)\xi_x$. Hence we have

$$\mathbb{E}[e^{i\tilde{Z}_n}|S] = \prod_{x\in\mathbb{Z}^d} \varphi_\xi\left(\frac{\sum_{i=1}^m \theta_i b_{i,n}(x)}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}}\right).$$

Observe next that

$$\left|\varphi_\xi(t) - \exp\left(-|t|^\beta(A_1 + iA_2\mathrm{sgn}(t)\right)\right| \le |t|^\beta h(|t|) \quad \text{for all } t\in\mathbb{R},$$

with h a continuous and monotone function on $[0,+\infty)$ vanishing in 0. This implies in particular the existence of $\varepsilon_0 > 0$ and $\sigma > 0$ such that $\max(|\varphi_\xi(t)|, \exp\left(-A_1|t|^\beta\right)) \le e^{-\sigma|t|^\beta}$ for any $t\in[-\varepsilon_0,\varepsilon_0]$. According to lemma 5, $\mathbb{P}$-almost surely, for every n large enough, we have

$$b_n := \sup_x \frac{|\sum_{i=1}^m \theta_i b_{i,n}(x)|}{n^{1/\beta}(\log(n))^{(\beta-1)/\beta}} \le \varepsilon_0$$

and so

$$\left|\mathbb{E}[e^{i\tilde{Z}_n}|S] - \prod_{x\in\mathbb{Z}^d} e^{-\frac{\left|\sum_{i=1}^m \theta_i b_{i,n}(x)\right|^\beta}{n(\log(n))^{\beta-1}}(A_1+iA_2\mathrm{sgn}\left(\sum_{i=1}^m \theta_i b_{i,n}(x)\right))}\right|$$

is less than $\sum_{x\in\mathbb{Z}^d}\frac{\left|\sum_{i=1}^m \theta_i b_{i,n}(x)\right|^\beta}{n(\log(n))^{\beta-1}} h(b_n) e^{-\sigma\left(\frac{\sum_{y\in\mathbb{Z}}\left|\sum_{i=1}^m \theta_i b_{i,n}(y)\right|^\beta}{n(\log n)^{\beta-1}} - b_n^\beta\right)}$. Hence, according to lemmas 4 and 5, $\mathbb{P}$-almost surely, we have

$$\lim_{n\to+\infty}\mathbb{E}[e^{i\tilde{Z}_n}|S] = e^{-\frac{\Gamma(\beta+1)}{(\pi A)^{\beta-1}}\sum_{i=1}^m |\theta_i|^\beta (t_i - t_{i-1})(A_1 + iA_2\mathrm{sgn}(\theta_i))}$$

which gives (9) thanks to the Lebesgue dominated convergence theorem.

Finally we prove that the sequence

$$\left(\left(\frac{Z_{[nt]}}{n^{1/\beta}\log(n)^{(\beta-1)/\beta}}\right)_{t\in[0;1]}\right)_{n\ge2}$$

is not tight in $\mathcal{D}([0,\infty))$. It is enough to prove that it is not tight in $\mathcal{D}([0,1])$. To this aim, let $b_n = n^{1/\beta}\log(n)^{(\beta-1)/\beta}$, and $(Z_n(t), t\in[0,1])$ denote the linear interpolation of $(Z_{[nt]}, t\in[0,1])$, i.e.

$$Z_n(t) = Z_{[nt]} + (nt - [nt])\xi_{S_{[nt]}}.$$

Then, $\forall\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}\left[\sup_{t\in[0,1]}\left|Z_n(t) - Z_{[nt]}\right| \ge \epsilon b_n\right] &= \mathbb{P}\left[\max_{i=0}^{n-1}|\xi_{S_i}| \ge \epsilon b_n\right]\\
&= \mathbb{P}\left[\exists x\in\{S_0,\cdots,S_{n-1}\} \text{ s.t } |\xi_x| \ge \epsilon b_n\right]\\
&\le \mathbb{E}(\#\{S_0,\cdots,S_{n-1}\})\mathbb{P}\left[|\xi_0| \ge \epsilon b_n\right]\\
&\le C\frac{n}{\log(n)}\epsilon^{-\beta}b_n^{-\beta} = C\epsilon^{-\beta}\log(n)^{-\beta},
\end{aligned}$$

where the last inequality comes from (2) and Theorem 6.9 of [13]. Therefore, if $\left(\left(\frac{Z_{[nt]}}{b_n}\right)_{t\in[0;1]}\right)_{n\ge2}$ converges weakly to $\left(\tilde{Y}_t\right)_{t\in[0,1]}$, the same is true for $\left(\left(\frac{Z_n(t)}{b_n}\right)_{t\in[0;1]}\right)_{n\ge2}$. Using the fact that the sequence $\left(\left(\frac{Z_n(t)}{b_n}\right)_{t\in[0;1]}\right)_{n\geq 2}$ is a sequence in the space $\mathbb{C}([0,1])$ and that the Skorohod J_1-topology coincides with the uniform one when restricted to $\mathbb{C}([0,1])$, one deduces that $\left(\frac{Z_n(t)}{b_n}\right)_{t\in[0;1]}$ converges weakly in $\mathbb{C}([0,1])$, and that the limiting process $\left(\tilde{Y}_t\right)_{t\in[0,1]}$ is therefore continuous, which is false as soon as $\beta < 2$. □

3. Proof of the local limit theorem in the lattice case

3.1. The event Ω_n.

Set

$$N_n^* := \sup_y N_n(y) \quad \text{and} \quad R_n := \#\{y \ : \ N_n(y) > 0\}.$$

Lemma 6. *For every $n \geq 1$ and $1 > \gamma > 0$, set*

$$\Omega_n = \Omega_n(\gamma) := \left\{R_n \leq \frac{n}{(\log\log(n))^{1/4}} \ \textit{and}\ N_n^* \leq n^\gamma\right\}.$$

Then, $\mathbb{P}(\Omega_n) = 1 - o(b_n^{-1})$. Moreover, the following also holds on Ω_n:

$$(\log\log(n))^{1/4} \leq N_n^* \quad \textit{and} \quad V_n \geq n^{1-\gamma(1-\beta)_+}. \tag{10}$$

Proof. We first prove that

$$\mathbb{P}\left(R_n \geq n(\log\log(n))^{-1/4}\right) = o(b_n^{-1}). \tag{11}$$

Let us recall that for every $a, b \in \mathbb{N}$, we have

$$\mathbb{P}(R_n \geq a+b) \leq \mathbb{P}(R_n \geq a)\mathbb{P}(R_n \geq b). \tag{12}$$

The proof is given for instance in [7]. We will moreover use the fact that $\mathbb{E}[R_n] \sim cn(\log(n))^{-1}$ and $Var(R_n) = O\left(n^2\log^{-4}(n)\right)$ (see [13]). Hence, for n large enough, there exists $C > 0$ such that we have

$$\begin{aligned}
\mathbb{P}\left(R_n \geq \frac{n}{(\log\log(n))^{1/4}}\right) &\leq \mathbb{P}\left(R_n \geq \left\lfloor \frac{n(\log\log(n))^{1/4}}{\log(n)} \right\rfloor\right)^{\left\lfloor \log(n)(\log\log(n))^{-1/2}\right\rfloor} \\
\leq\ & \mathbb{P}\left(|R_n - \mathbb{E}[R_n]| \geq \frac{1}{2}\left\lfloor \frac{n(\log\log(n))^{1/4}}{\log(n)} \right\rfloor\right)^{\left\lfloor \log(n)(\log\log(n))^{-1/2}\right\rfloor} \\
\leq\ & \left(\frac{5Var(R_n)\log^2(n)}{n^2(\log\log(n))^{1/2}}\right)^{\left\lfloor \log(n)(\log\log(n))^{-1/2}\right\rfloor} \\
\leq\ & \left(\frac{Cn^2\log^2(n)/\log^4(n)}{n^2\sqrt{\log\log(n)}}\right)^{\left\lfloor \log(n)(\log\log(n))^{-1/2}\right\rfloor} \\
\leq\ & \left(\frac{C}{(\log(n))^2}\right)^{\left\lfloor \log(n)(\log\log(n))^{-1/2}\right\rfloor} = \exp\left(-\log(n)\sqrt{\log\log(n)}\left(1 - \frac{\log(C)}{2\log\log(n)}\right)\right).
\end{aligned}$$

This ends the proof of (11).

Let us now prove that

$$\mathbb{P}\left[N_n^* \geq n^\gamma\right] = o(b_n^{-1}). \tag{13}$$

We have

$$\begin{aligned}
\mathbb{P}(N_n^* \geq n^\gamma) &\leq \sum_x \mathbb{P}(N_n(x) \geq n^\gamma)\\
&= \sum_x \mathbb{P}(T_x \leq n; N_n(x) \geq n^\gamma)\,, \text{ where } T_x := \inf\{n > 1, \text{ s.t. } S_n = x\}\,,\\
&\leq \sum_x \mathbb{P}(T_x \leq n)\mathbb{P}(N_n(0) \geq n^\gamma)\\
&\leq \mathbb{E}[R_n]\mathbb{P}(T_0 \leq n)^{n^\gamma}.
\end{aligned}$$

Hence, (13) follows now from $\mathbb{E}[R_n] \sim cn(log(n))^{-1}$, and from $\mathbb{P}(T_0 > n) \sim C/\log(n)$.

Since $n = \sum_y N_n(y) \leq R_n N_n^*$, we get that $N_n^* \geq \frac{n}{R_n} \geq (\log\log(n))^{1/4}$ on Ω_n.

To prove the lower bound for V_n, note that for $\beta \geq 1$, $V_n = \sum_y N_n(y)^\beta \geq \sum_y N_n(y) = n$. For $\beta < 1$, on Ω_n,

$$n = \sum_y N_n(y) = \sum_y N_n(y)^\beta N_n(y)^{1-\beta} \leq V_n(N_n^*)^{1-\beta} \leq V_n n^{\gamma(1-\beta)}\,.$$

□

3.2. **Scheme of the proof.** It is easy to see (cf the proof of lemma 5 in [5]) that $\mathbb{P}(Z_n = \lfloor b_n x\rfloor) = 0$ if $\mathbb{P}(n\xi_0 - \lfloor b_n x\rfloor \notin d_0\mathbb{Z}) = 1$, and that if $\mathbb{P}(n\xi_0 - \lfloor b_n x\rfloor \in d_0\mathbb{Z}) = 1$,

$$\mathbb{P}(Z_n = \lfloor b_n x\rfloor) = \frac{d_0}{2\pi}\int_{-\frac{\pi}{d_0}}^{\frac{\pi}{d_0}} e^{-it\lfloor b_n x\rfloor}\mathbb{E}\left[\prod_y \varphi_\xi(tN_n(y))\right] dt\,.$$

In view of lemma 6, we have to estimate

$$\frac{d_0}{2\pi}\int_{-\frac{\pi}{d_0}}^{\frac{\pi}{d_0}} e^{-it\lfloor b_n x\rfloor}\mathbb{E}\left[\prod_y \varphi_\xi(tN_n(y))\mathbf{1}_{\Omega_n}\right] dt\,.$$

This is done in several steps presented in the following propositions.

Proposition 7. *Let $\gamma \in (0, 1/(\beta+1))$ and $\delta \in (0, 1/(2\beta))$ s.t. $\gamma\frac{(1-\beta)_+}{\beta} < \delta < 1/\beta - \gamma$. Then, we have*

$$\frac{d_0}{2\pi}\int_{\{|t|\leq n^\delta/b_n\}} e^{-it\lfloor b_n x\rfloor}\mathbb{E}\left[\prod_y \varphi_\xi(tN_n(y))\mathbf{1}_{\Omega_n}\right] dt = d_0\frac{C(x)}{b_n} + o(b_n^{-1})\,,$$

uniformly in $x \in \mathbb{R}$.

Recall next that the characteristic function ϕ of the limit distribution of $\left(n^{-1/\beta}\sum_{k=1}^n \xi_{ke_1}\right)_n$ has the following form :

$$\phi(u) = e^{-|u|^\beta(A_1 + iA_2 sgn(u))},$$

with $0 < A_1 < \infty$ and $|A_1^{-1}A_2| \leq |\tan(\pi\beta/2)|$. It follows that the characteristic function φ_ξ of ξ_0 satisfies:

$$1 - \varphi_\xi(u) \sim |u|^\beta(A_1 + iA_2 sgn(u)) \quad \text{when } u \to 0. \tag{14}$$

Therefore there exist constants $\varepsilon_0 > 0$ and $\sigma > 0$ such that

$$\max(|\phi(u)|, |\varphi_\xi(u)|) \leq \exp\left(-\sigma|u|^\beta\right) \quad \text{for all } u \in [-\varepsilon_0, \varepsilon_0]. \tag{15}$$

Since $\overline{\varphi_\xi(t)} = \varphi_\xi(-t)$ for every $t \geq 0$, the following propositions achieve the proof of Theorem 2:

Proposition 8. *Let δ and γ be as in Proposition 7. Then there exists $c>0$ such that*

$$\int_{n^\delta/b_n}^{\varepsilon_0 n^{-\gamma}} \mathbb{E}\left[\prod_y |\varphi_\xi(tN_n(y))|\mathbf{1}_{\Omega_n}\right] dt = o(e^{-n^c}).$$

Proposition 9. *There exists $c>0$ such that*

$$\int_{\varepsilon_0 n^{-\gamma}}^{\frac{\pi}{d_0}} \mathbb{E}\left[\prod_y |\varphi_\xi(tN_n(y))|\mathbf{1}_{\Omega_n}\right] dt = o(e^{-n^c}).$$

3.3. **Proof of Proposition 7.** Remember that $V_n=\sum_{z\in\mathbb{Z}^d} N_n^\beta(z)$. We start by a preliminary lemma.

Lemma 10. (1) *If $\beta>1$, $\sup_n \mathbb{E}\left[\left(\frac{n\log(n)^{\beta-1}}{V_n}\right)^{1/(\beta-1)}\right]<+\infty$.*

(2) *If $\beta\le 1$, $\forall p\in\mathbb{N}$, $\sup_n \mathbb{E}\left[\left(\frac{n\log(n)^{\beta-1}}{V_n}\right)^{p}\right]<+\infty$.*

Proof. For $\beta>1$, using Hölder's inequality with $p=\beta$, we get

$$n=\sum_x N_n(x)\le V_n^{\frac{1}{\beta}} R_n^{\frac{\beta-1}{\beta}}$$

which means that

$$\left(\frac{n\log(n)^{\beta-1}}{V_n}\right)^{1/(\beta-1)} \le \frac{\log(n)R_n}{n}.$$

But it is proved in [13] Equation (7.a) that $\mathbb{E}[R_n]=\mathcal{O}(n/\log(n))$. The result follows.

The result is obvious for $\beta=1$. For $\beta<1$, Hölder's inequality with $p=2-\beta$ yields

$$n=\sum_x N_n^{\frac{\beta}{2-\beta}}(x)N_n^{\frac{2(1-\beta)}{2-\beta}}(x)\le V_n^{\frac{1}{2-\beta}}\left(\sum_x N_n^2(x)\right)^{\frac{1-\beta}{2-\beta}}$$

and so

$$\frac{n\log(n)^{\beta-1}}{V_n}\le\left(\frac{\sum_x N_n^2(x)}{n\log(n)}\right)^{1-\beta}.$$

It is therefore enough to prove that there exists $c>0$ such that

$$\sup_n \mathbb{E}\left[\exp\left(c\frac{\sum_x N_n^2(x)}{n\log(n)}\right)\right]<\infty. \tag{16}$$

Note that $\sum_x N_n^2(x)=\sum_{k=0}^{n-1}N_n(S_k)$. By Jensen's inequality, we get thus

$$\mathbb{E}\left[\exp\left(c\frac{\sum_x N_n^2(x)}{n\log(n)}\right)\right]\le\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{E}\left[\exp\left(c\frac{N_n(S_k)}{\log(n)}\right)\right].$$

Observe now that $N_n(S_k)=\sum_{j=0}^k \mathbf{1}_{\{S_k-S_j=0\}}+\sum_{j=k+1}^{n-1}\mathbf{1}_{\{S_j-S_k=0\}}\overset{(d)}{=}N_{k+1}(0)+N'_{n-k}(0)-1$, where $(N'_n(x),n\in\mathbb{N},x\in\mathbb{Z}^d)$ is an independent copy of $(N_n(x),n\in\mathbb{N},x\in\mathbb{Z}^d)$. Hence,

$$\mathbb{E}\left[\exp\left(c\frac{\sum_x N_n^2(x)}{n\log(n)}\right)\right]\le\mathbb{E}\left[\exp\left(c\frac{N_n(0)}{\log(n)}\right)\right]^2.$$

But, $\forall t>0$,

$$\mathbb{P}\left(N_n(0)\ge t\log(n)\right)\le\mathbb{P}\left(T_0\le n\right)^{\lceil t\log(n)\rceil},$$

and
$$\mathbb{E}\left[\exp\left(c\frac{N_n(0)}{\log(n)}\right)\right] \le 1 + \int_0^\infty c\exp(ct)\exp\left(-\lceil t\log(n)\rceil\,\mathbb{P}(T_0 > n)\right)\,dt\,.$$
Now (16) follows then from the fact that $\exists C>0$ such that $\mathbb{P}(T_0>n)\sim C/\log(n)$ for any integer $n\ge 1$. $\square$

The next step is

Lemma 11. *Under the hypotheses of Proposition 7, we have*
$$\int_{\{|t|\le n^\delta/b_n\}} e^{-it\lfloor b_n x\rfloor}\mathbb{E}\left[\left\{\prod_y \varphi_\xi(tN_n(y)) - e^{-|t|^\beta(A_1+iA_2 sgn(t))V_n}\right\}\mathbf{1}_{\Omega_n}\right]dt = o(b_n^{-1})\,,$$
uniformly in $x\in\mathbb{R}$.

Proof. It suffices to prove that
$$\int_{\{|t|\le n^\delta/b_n\}}\mathbb{E}[|E_n(t)|\mathbf{1}_{\Omega_n}]\,dt = o(b_n^{-1})$$
with
$$E_n(t) := \prod_y \varphi_\xi(tN_n(y)) - \prod_y \exp\left(-|t|^\beta N_n^\beta(y)(A_1+iA_2 sgn(t))\right).$$
Observe that
$$\begin{aligned} E_n(t) = \sum_y \quad & \left(\prod_{z<y}\varphi_\xi(tN_n(z))\right)\left(\varphi_\xi(tN_n(y)) - e^{-|t|^\beta N_n^\beta(y)(A_1+iA_2 sgn(t))}\right)\\ \times\ & \left(\prod_{z>y} e^{-|t|^\beta N_n^\beta(z)(A_1+iA_2 sgn(t))}\right), \end{aligned}$$
where an arbitrary ordering of sites of $\mathbb{Z}^d$ has been chosen. But on Ω_n, if $|t|\le n^\delta b_n^{-1}$, then
$$|t|N_n(z) \le n^{\gamma+\delta}b_n^{-1}. \tag{17}$$
Since $\gamma+\delta<\beta^{-1}$, this implies in particular that $|t|N_n(z)<\varepsilon_0$ for n large enough. Thus, by using (15), we get
$$|E_n(t)| \le \sum_y \left|\varphi_\xi(tN_n(y)) - \exp\left(-|t|^\beta N_n^\beta(y)(A_1+iA_2 sgn(t))\right)\right|\exp\left(-\sigma|t|^\beta\sum_{z\ne y}N_n^\beta(z)\right),$$
for n large enough. Observe next that (14) implies
$$\left|\varphi_\xi(u) - \exp\left(-|u|^\beta(A_1+iA_2 sgn(u))\right)\right| \le |u|^\beta h(|u|) \quad \text{for all } u\in\mathbb{R},$$
with h a continuous and monotone function on $[0,+\infty)$ vanishing in 0. Therefore by using (17) we get
$$|E_n(t)| \le |t|^\beta h(n^{\gamma+\delta}b_n^{-1})\sum_y N_n^\beta(y)\exp\left(-\sigma|t|^\beta\sum_{z\ne y}N_n^\beta(z)\right).$$
Now, according to (10) and since $\gamma<\frac{1}{\beta+1}\le\frac{1}{\beta+(1-\beta)_+}$, if n is large enough, we have on Ω_n
$$\sum_{z\ne y}N_n^\beta(z) \ge V_n/2 \qquad \text{for all } y\in\mathbb{Z}.$$

By using this and the change of variables $v = tV_n^{1/\beta}$, we get

$$\int_{\{|t|\le n^\delta b_n^{-1}\}} \mathbb{E}\left[|E_n(t)|\mathbf{1}_{\Omega_n}\right] dt \le h(n^{\gamma+\delta}b_n^{-1})\mathbb{E}[V_n^{-1/\beta}]\int_{\mathbb{R}} |v|^\beta \exp\left(-\sigma|v|^\beta/2\right) dv = o(\mathbb{E}[V_n^{-1/\beta}]),$$

which proves the result according to Lemma 10. □

Finally Proposition 7 follows from the

Lemma 12. *Under the hypotheses of Proposition 7, we have*

$$\frac{d_0}{2\pi}\int_{\{|t|\le n^\delta b_n^{-1}\}} e^{-it\lfloor b_n x\rfloor}\mathbb{E}\left[e^{-|t|^\beta V_n(A_1+iA_2 sgn(t))}\mathbf{1}_{\Omega_n}\right] dt = d_0\frac{C(x)}{b_n} + o(b_n^{-1}),$$

uniformly in $x \in \mathbb{R}$.

Proof. Set

$$I_{n,x} := \int_{\{|t|\le n^\delta b_n^{-1}\}} e^{-it\lfloor b_n x\rfloor} e^{-|t|^\beta V_n(A_1+iA_2 sgn(t))}\, dt,$$

which can be rewritten

$$I_{n,x} = \int_{\{|t|\le n^\delta b_n^{-1}\}} e^{-it\lfloor b_n x\rfloor}\phi(tV_n^{1/\beta})\, dt.$$

Since $|\lfloor b_n x\rfloor - b_n x| \le 1$, for all n and x, it is immediate that

$$I_{n,x} = \int_{\{|t|\le n^\delta b_n^{-1}\}} e^{-itb_n x}\phi(tV_n^{1/\beta})\, dt + \mathcal{O}(n^{2\delta}b_n^{-2}).$$

But $\delta < (2\beta)^{-1}$ by hypothesis. So actually

$$I_{n,x} = \int_{\{|t|\le n^\delta b_n^{-1}\}} e^{-itb_n x}\phi(tV_n^{1/\beta})\, dt + o(b_n^{-1}).$$

Next, with the change of variable $v = tb_n$, we get:

$$\int_{\{|t|\le n^\delta b_n^{-1}\}} e^{-itb_n x}\phi(tV_n^{1/\beta})\, dt = b_n^{-1}\left\{V_n^{-1/\beta}b_n f(xV_n^{-1/\beta}b_n) - J_{n,x}\right\}, \tag{18}$$

where f is the density function of the distribution with characteristic function ϕ and where

$$J_{n,x} := \int_{\{|v|\ge n^\delta\}} e^{-ivx}\phi(vb_n^{-1}V_n^{1/\beta})\, dv.$$

By lemma 4 (applied with $m = 1$, $t_1 = \theta_1 = 1$, $\gamma = \beta$), $(W_n := b_n V_n^{-1/\beta})_n$ converges almost surely, as $n \to \infty$, to the constant $\Gamma(\beta+1)^{-1/\beta}(\pi A)^{1-1/\beta}$. Moreover, Lemma 10 ensures that the sequence $(W_n, n \ge 1)$ is uniformly integrable, so actually the convergence holds in $\mathbb{L}^1$. Let us deduce that

$$\mathbb{E}[g_x(W_n)] = \mathbb{E}[g_x(W)] + o(1), \tag{19}$$

where $g_x : z \mapsto zf(xz)$ and the $o(1)$ is uniform in x. First

$$\begin{aligned} |\mathbb{E}[g_x(W_n)] - \mathbb{E}[g_x(W)]| &\le \sup_{x,z\in\mathbb{R}} |(g_x)'(z)|\mathbb{E}[|W_n - W|] \\ &\le \sup_u |f(u) + uf'(u)|\mathbb{E}[|W_n - W|]. \end{aligned}$$

This proves (19). We observe that $\mathbb{E}[g_x(W)] = C(x)$.

In view of (18), it only remains to prove that $\mathbb{E}[J_{n,x}\mathbf{1}_{\Omega_n}] = o(1)$ uniformly in x. But this follows from the basic inequality

$$\mathbb{E}[|J_{n,x}\mathbf{1}_{\Omega_n}|] \leq \int_{|v|\geq n^\delta} \mathbb{E}\left[e^{-A_1|v|^\beta \frac{V_n}{b_n^\beta}}\mathbf{1}_{\Omega_n}\right] dv,$$

and from the lower bound for V_n given in (10) and from the choice $\delta > \gamma(1-\beta)_+/\beta$. □

3.4. **Proof of Proposition 8.** Recall that on Ω_n, $N_n(y) \leq n^\gamma$, for all $y \in \mathbb{Z}^d$. Hence by (15),

$$K_n := \int_{n^\delta/b_n}^{\varepsilon_0 n^{-\gamma}} \mathbb{E}\left[\prod_y |\varphi_\xi(tN_n(y))|\mathbf{1}_{\Omega_n}\right] dt \leq \int_{n^\delta/b_n}^{\varepsilon_0 n^{-\gamma}} \mathbb{E}\left[\exp\left(-\sigma t^\beta V_n\right)\mathbf{1}_{\Omega_n}\right] dt\,.$$

With the change of variable $s = tV_n^{1/\beta}$, we get

$$\begin{aligned} K_n &\leq \mathbb{E}\left[V_n^{-1/\beta}\int_{n^\delta V_n^{1/\beta} b_n^{-1}}^{\varepsilon_0 n^{-\gamma}V_n^{1/\beta}} \exp\left(-\sigma s^\beta\right) ds\mathbf{1}_{\Omega_n}\right] \\ &\leq \frac{1}{n^{\frac{1}{\beta}-\gamma\frac{(1-\beta)_+}{\beta}}}\int_{n^{\delta-\gamma\frac{(1-\beta)_+}{\beta}}\log(n)^{\frac{1-\beta}{\beta}}}^{+\infty} \exp\left(-\sigma s^\beta\right) ds\,, \end{aligned}$$

which proves the proposition since $\delta > \gamma(1-\beta)_+/\beta$.

3.5. **Proof of Proposition 9.** We adapt the proof of [5, Proposition 10]. We will see that the argument of "peaks" still works here. We endow $\mathbb{Z}^d$ with the ordered structure given by the relation $<$ defined by

$$(\alpha_1, ..., \alpha_d) < (\beta_1, ..., \beta_d) \ \leftrightarrow\ \exists i \in \{1, ..., d\},\quad \alpha_i < \beta_i,\ \forall j < i,\ \alpha_j = \beta_j.$$

We consider $\mathcal{C}^+ = (x_1, ..., x_T) \in (\mathbb{Z}^d \setminus \{0\})^T$ for some positive integer T such that:

- $x_1 + ... + x_T = 0$;
- for every $i = 1, ..., T$, $\mathbb{P}(X_1 = x_i) > 0$;
- there exists $I_1 \in \{1, ..., T\}$ such that
  - for every $i = 1, ..., I_1$, $x_i > 0$,
  - for every $i = I_1 + 1, ..., T$, $x_i < 0$.

Let us write $\mathcal{C}^- := (x_{T-i+1})_{i=1,...,T}$. We define $B := \sum_{i=1}^{I_1} x_i$. We observe that

$$p := \mathbb{P}((X_1, ..., X_T) = \mathcal{C}^+) = \mathbb{P}((X_1, ..., X_T) = \mathcal{C}^-) > 0.$$

We notice that $(X_1, ..., X_T) = \mathcal{C}^+$ corresponds to a trajectory visiting B only once before going back to the origin at time T (and without visiting $-B$). Analogously, $(X_1, ..., X_T) = \mathcal{C}^-$ corresponds to a trajectory that goes down to $-B$ and comes back up to 0 (and without visiting B), and staying at a distance smaller than $\tilde{d}/2$ of the origin with $\tilde{d} := \sum_{i=1}^T |x_i|$ (where $|\cdot|$ is the absolute value if $d = 1$ and $|(a, b)| = \max(|a|, |b|)$ if $d = 2$). We introduce now the event

$$\mathcal{D}_n := \left\{C_n > \frac{np}{2T}\right\},$$

where

$$C_n := \#\left\{k = 0, ..., \left\lfloor\frac{n}{T}\right\rfloor - 1\ :\ (X_{kT+1}, \dots, X_{(k+1)T}) = \mathcal{C}^\pm\right\}.$$

Since the sequences $(X_{kT+1}, \dots, X_{(k+1)T})$, for $k \geq 0$, are independent of each other, Chernoff's inequality implies that there exists $c > 0$ such that

$$\mathbb{P}(\mathcal{D}_n) = 1 - o(e^{-cn}).$$

We introduce now the notion of "loop". We say that there is a loop based on y at time n if $S_n = y$ and $(X_{n+1}, \dots, X_{n+T}) = \mathcal{C}^{\pm}$. We will see (in Lemma 13 below) that, on $\Omega_n \cap \mathcal{D}_n$, there is a large number of $y \in \mathbb{Z}^d$ on which are based a large number of loops. For any $y \in \mathbb{Z}^d$, let

$$C_n(y) := \#\left\{k = 0, \dots, \left\lfloor \frac{n}{T} \right\rfloor - 1 \;:\; S_{kT} = y \text{ and } (X_{kT+1}, \dots, X_{(k+1)T}) = \mathcal{C}^{\pm}\right\},$$

be the number of loops based on y before time n (and at times which are multiple of T), and let

$$p_n := \#\left\{ y \in \mathbb{Z} \;:\; C_n(y) \geq \frac{\log\log(n)^{1/4} p}{4T} \right\},$$

be the number of sites $y \in \mathbb{Z}$ on which at least $a_n := \left\lfloor \frac{\log\log(n)^{1/4} p}{4T} \right\rfloor$ loops are based.

Lemma 13. *On $\Omega_n \cap \mathcal{D}_n$, we have, $p_n \geq c' n^{1-\gamma}$ with $c' = p/(4T)$.*

Proof. Note that $C_n(y) \leq N_n^*$ for all $y \in \mathbb{Z}^d$. Thus on $\Omega_n \cap \mathcal{D}_n$, we have

$$\begin{aligned} \frac{np}{2T} &\leq \sum_{y \in \mathbb{Z}^d \,:\, C_n(y) < a_n} C_n(y) + \sum_{y \in \mathbb{Z}^d \,:\, C_n(y) \geq a_n} C_n(y) \\ &\leq R_n a_n + N_n^* p_n \leq \frac{np}{4T} + p_n n^{\gamma}, \end{aligned}$$

according to lemma 6. This proves the lemma. □

We have proved that, if n is large enough, the event $\Omega_n \cap \mathcal{D}_n$ is contained in the event

$$\mathcal{E}_n := \{p_n \geq c' n^{1-\gamma}\}.$$

Now, on $\mathcal{E}_n$, we consider $(Y_i)_{i=1,\dots,\lfloor c'' n^{1-\gamma} \rfloor}$ (with $c'' := c'/(2\tilde{d})$ if $d = 1$ and with $c'' := c'/2\tilde{d}^2$) if $d = 2$) such that

- on each Y_i, at least a_n loops are based,
- for every i, j such that $i \neq j$, we have $|Y_i - Y_j| > \tilde{d}/2$.

For every $i = 1, \dots, \lfloor c'' n^{1-\gamma} \rfloor$, let $t_i^{(1)}, \dots, t_i^{(a_n)}$ be the a_n first times (which are multiples of T) when a loop is based on the site Y_i. We also define $N_n^0(Y_i + B)$ as the number of visits of S before time n to $Y_i + B$, which do not occur during the time intervals $[t_i^{(j)}, t_i^{(j)} + T]$, for $j \leq a_n$.

Since our construction is basically the same as in [5, section 2.8], the proof of the following lemma is exactly the same as the proof of [5, Lemma 16] and we do not prove it again.

Lemma 14. *Conditionally to the event $\mathcal{E}_n$, $(N_n(Y_i + B) - N_n^0(Y_i + B))_{i \geq 1}$ is a sequence of independent identically distributed random variables with binomial distribution $\mathcal{B}\left(a_n; \frac{1}{2}\right)$. Moreover this sequence is independent of $(N_n^0(Y_i + B))_{i \geq 1}$.*

Let η be a real number such that $\gamma < \eta < (1-\gamma)/\beta$ (this is possible since $\gamma < 1/(\beta+1)$). We define

$$\forall n \geq 1, \quad d_n := n^{-\eta}.$$

Let now $\rho := \sup\{|\varphi_\xi(u)| \;:\; d\left(u, \frac{2\pi}{d_0}\mathbb{Z}\right) \geq \varepsilon_0\}$. According to Formula (15) and since $\lim_{n\to\infty} d_n = 0$, for n large enough, we have

$$\begin{aligned} |\varphi_\xi(u)| &\leq \rho \mathbf{1}_{\{d\left(u, \frac{2\pi}{d_0}\mathbb{Z}\right) \geq \epsilon_0\}} + \exp\left(-\sigma d\left(u, \frac{2\pi}{d_0}\mathbb{Z}\right)^{\beta}\right) \mathbf{1}_{\{d\left(u, \frac{2\pi}{d_0}\mathbb{Z}\right) < \epsilon_0\}} \\ &\leq \exp\left(-\sigma d_n^{\beta}\right), \end{aligned}$$

as soon as $d\left(u, \frac{2\pi}{d_0}\mathbb{Z}\right) \geq d_n$. Therefore, for n large enough,

$$\prod_z |\varphi_\xi(tN_n(z))| \leq \exp\left(-\sigma d_n^\beta \# \left\{z \ : \ d\left(tN_n(z), \frac{2\pi}{d_0}\mathbb{Z}\right) \geq d_n\right\}\right). \tag{20}$$

Then notice that

$$d\left(tN_n(z), \frac{2\pi\mathbb{Z}}{d_0}\right) \geq d_n \iff N_n(z) \in \mathcal{I} := \bigcup_{k\in\mathbb{Z}} I_k, \tag{21}$$

where for all $k \in \mathbb{Z}$,

$$I_k := \left[\frac{2k\pi}{d_0 t} + \frac{d_n}{t}, \frac{2(k+1)\pi}{d_0 t} - \frac{d_n}{t}\right].$$

In particular $\mathbb{R} \setminus \mathcal{I} = \bigcup_{k\in\mathbb{Z}} J_k$, where for all $k \in \mathbb{Z}$,

$$J_k := \left(\frac{2k\pi}{d_0 t} - \frac{d_n}{t}, \frac{2k\pi}{d_0 t} + \frac{d_n}{t}\right).$$

Lemma 15. *Under the hypotheses of Proposition 9, for every $i \leq \lfloor c'' n^{1-\gamma} \rfloor$, $t \in (\varepsilon_0 n^{-\gamma}, \pi/d_0)$ and n large enough,*

$$\mathbb{P}\left(N_n(Y_i + B) \in \mathcal{I} \mid \mathcal{E}_n, \ N_n^0(Y_i + B)\right) \geq \frac{1}{3} \quad \textit{almost surely.}$$

Assume for a moment that this lemma holds true and let us finish the proof of Proposition 9. Lemmas 14 and 15 ensure that conditionally to $\mathcal{E}_n$ and $((N_n^0(Y_i + B), i \geq 1)$, the events $\{N_n(Y_i + B) \in \mathcal{I}\}$, $i \geq 1$, are independent of each other, and all happen with probability at least 1/3. Therefore, since $\Omega_n \cap \mathcal{D}_n \subseteq \mathcal{E}_n$, there exists $c > 0$, such that

$$\mathbb{P}\left(\Omega_n \cap \mathcal{D}_n, \ \#\{i \ : \ N_n(Y_i + B) \in \mathcal{I}\} \leq \frac{c'' n^{1-\gamma}}{4}\right) \leq \mathbb{P}\left(B_n \leq \frac{c'' n^{1-\gamma}}{4}\right) = o(\exp(-cn^{1-\gamma})),$$

where for all $n \geq 1$, B_n has binomial distribution $\mathcal{B}\left(\lfloor c'' n^{1-\gamma} \rfloor ; \frac{1}{3}\right)$.

But if $\#\{z \ : \ N_n(z) \in \mathcal{I}\} \geq \frac{c'' n^{1-\gamma}}{4}$, then by (20) and (21) there exists a constant $c > 0$, such that

$$\prod_z |\varphi_\xi(tN_n(z))| \leq \exp\left(-cn^{1-\gamma} d_n^\beta\right),$$

which proves Proposition 9 since $1 - \gamma - \beta\eta > 0$.

Proof of Lemma 15. First notice that by Lemma 14, for any $H \geq 0$,

$$\mathbb{P}(N_n(Y_i + B) \in \mathcal{I} \mid \mathcal{E}_n, \ N_n^0(Y_i + B) = H) = \mathbb{P}\left(H + b_n \in \mathcal{I}\right), \tag{22}$$

where b_n is a random variable with binomial distribution $\mathcal{B}\left(a_n; \frac{1}{2}\right)$. We will use the following result whose proof is postponed.

Lemma 16. *Under the hypotheses of Proposition 9, for every $t \in (\varepsilon_0 n^{-\gamma}, \pi/d_0)$ and for n large enough, the following holds:*

(i) *For any integer k such that all the elements of $I_k - H$ are smaller than $\frac{a_n}{2}$,*

$$\mathbb{P}(b_n \in (I_k - H)) \geq \mathbb{P}(b_n \in (J_k - H)).$$

(ii) *For any integer k such that all the elements of $I_k - H$ are larger than $\frac{a_n}{2}$,*

$$\mathbb{P}(b_n \in (I_k - H)) \geq \mathbb{P}(b_n \in (J_{k+1} - H)).$$

Now call k_0 the largest integer satisfying the condition appearing in (i) and k_1 the smallest integer satisfying the condition appearing in (ii). We have $k_1 = k_0 + 1$ or $k_1 = k_0 + 2$. According to Lemma 16, we have

$$\begin{aligned} \mathbb{P}\left(H + b_n \in \mathcal{I}\right) &\geq \sum_{k \leq k_0} \mathbb{P}\left(H + b_n \in I_k\right) + \sum_{k \geq k_1} \mathbb{P}\left(H + b_n \in I_k\right) \\ &\geq \sum_{k \leq k_0} \mathbb{P}\left(H + b_n \in J_k\right) + \sum_{k \geq k_1} \mathbb{P}\left(H + b_n \in J_{k+1}\right) \\ &= \mathbb{P}(H + b_n \notin \mathcal{I}) - \mathbb{P}(H + b_n \in J_{k_0+1} \cup J_{k_1}). \end{aligned}$$

Hence,

$$\mathbb{P}\left(H + b_n \in \mathcal{I}\right) \geq \frac{1}{2}\left[1 - \mathbb{P}(H + b_n \in J_{k_0+1} \cup J_{k_1})\right].$$

Let $\bar{b}_n := 2\left(b_n - \frac{a_n}{2}\right)\sqrt{a_n}$. Since $\lim_{n\to+\infty} a_n = +\infty$, $(\bar{b}_n)_n$ converges in distribution to a standard normal variable, whose distribution function is denoted by Φ. The interval J_{k_1} being of length $2d_n/t$,

$$\begin{aligned} \mathbb{P}(H + b_n \in J_{k_1}) &= \mathbb{P}(\bar{b}_n \in [m_n, M_n])\,, \text{ with } M_n - m_n = 4\frac{d_n}{t\sqrt{a_n}} \\ &\leq \Phi(M_n) - \Phi(m_n) + \frac{C}{\sqrt{a_n}} \text{ (by the Berry–Esseen inequality)} \\ &\leq \frac{M_n - m_n}{\sqrt{2\pi}} + \frac{C}{\sqrt{a_n}} \\ &\leq C'\frac{d_n}{\varepsilon_0 n^{-\gamma}\sqrt{a_n}} + \frac{C}{\sqrt{a_n}}\,, \end{aligned}$$

for $t \geq \varepsilon_0 n^{-\gamma}$, and some constants $C > 0$ and $C' > 0$. Since $\lim_{n\to+\infty} a_n = +\infty$ and $\lim_{n\to+\infty} d_n n^{\gamma}(a_n)^{-1/2} = 0$ (since $\eta > \gamma$), we conclude that $\mathbb{P}(H + b_n \in J_{k_1}) = o(1)$. The same holds for $\mathbb{P}(H + b_n \in J_{k_0+1})$, so that for n large enough,

$$\mathbb{P}\left(H + b_n \in \mathcal{I}\right) \geq \frac{1}{2}\left[1 - o(1)\right] \geq \frac{1}{3}\,.$$

Together with (22), this concludes the proof of Lemma 15. □

Proof of Lemma 16. We only prove (i), since (ii) is similar. So let k be an integer such that all the elements of $I_k - H$ are smaller than $\frac{a_n}{2}$. Assume that $(J_k - H) \cap \mathbb{Z}$ contains at least one nonnegative integer (otherwise $\mathbb{P}(b_n \in (J_k - H)) = 0$ and there is nothing to prove). Let z_k denote the greatest integer in $J_k - H$, so that by our assumption $\mathbb{P}(b_n = z_k) > 0$ (remind that $0 \leq z_k < \frac{a_n}{2}$). By monotonicity of the function $z \mapsto \mathbb{P}(b_n = z)$, for $z \leq \frac{a_n}{2}$, we get

$$\mathbb{P}(b_n \in J_k - H) \leq \mathbb{P}(b_n = z_k)\#((J_k - H) \cap \mathbb{Z}) \leq \mathbb{P}(b_n = z_k)\left\lceil \frac{2d_n}{t} \right\rceil.$$

In the same way,

$$\mathbb{P}(b_n \in I_k - H) \geq \mathbb{P}(b_n = z_k)\#((I_k - H) \cap \mathbb{Z}) \geq \mathbb{P}(b_n = z_k)\left\lfloor \frac{2\pi}{d_0 t} - \frac{2d_n}{t} \right\rfloor.$$

Hence

$$\mathbb{P}(b_n \in I_k - H) \geq \frac{\left\lfloor \frac{2\pi}{d_0 t} - \frac{2d_n}{t} \right\rfloor}{\left\lceil \frac{2d_n}{t} \right\rceil}\mathbb{P}(b_n \in J_k - H)\,.$$

But $\pi/(d_0 t) \geq 1$ and $\lim_{n\to+\infty} d_n = 0$ by hypothesis. It follows immediately that for n large enough, we have $2d_n < \pi/(2d_0)$, and so

$$\left\lfloor \frac{2\pi}{d_0 t} - \frac{2d_n}{t} \right\rfloor \geq \left\lfloor \frac{3\pi}{2d_0 t} \right\rfloor \geq 1 + \left\lfloor \frac{\pi}{2d_0 t} \right\rfloor \geq \left\lceil \frac{\pi}{2d_0 t} \right\rceil \geq \left\lceil \frac{2d_n}{t} \right\rceil .$$

This concludes the proof of the lemma. □

4. Proof of the local limit theorem in the strongly nonlattice case

As in [5], the proof in the strongly nonlattice case is closely related to the proof in the lattice case.

We assume here that ξ is strongly nonlattice. In that case, there exist $\varepsilon_0 > 0$, $\sigma > 0$ and $\rho < 1$ such that $|\varphi_\xi(u)| \leq \rho$ if $|u| \geq \varepsilon_0$ and $|\varphi_\xi(u)| \leq \exp(-\sigma|u|^\beta)$ if $|u| < \varepsilon_0$.

We use here the notations of Section 3 with the hypotheses on γ, and δ of Proposition 7. Let h_0 be the density of Polya's distribution: $h_0(y) = \frac{1}{\pi}\frac{1-\cos(y)}{y^2}$, with Fourier transform $\hat{h}_0(t) = (1-|t|)_+$. For $\theta \in \mathbb{R}$, let $h_\theta(y) = \exp(i\theta y)h_0(y)$ with Fourier transform $\hat{h}_\theta(t) = \hat{h}_0(t+\theta)$. As in [10, thm 5.4], it is enough to show that for all $\theta \in \mathbb{R}$,

$$\lim_{n\to\infty} b_n \mathbb{E}\left[h_\theta(Z_n - b_n x)\right] = C(x)\,\hat{h}_\theta(0)\,. \tag{23}$$

By Fourier inverse transform, we have

$$b_n \mathbb{E}\left[h_\theta(Z_n - b_n x)\right] = \frac{b_n}{2\pi}\int_{\mathbb{R}} e^{-iub_n x}\mathbb{E}\left[\prod_{x\in\mathbb{Z}^d}\varphi_\xi(uN_n(x))\right]\hat{h}_\theta(u)\,du\,.$$

Since $\hat{h}_\theta \in L^1$, we can restrict our study to the event Ω_n of Lemma 6. The part of the integral corresponding to $|u| \leq n^\delta b_n^{-1}$ is treated exactly as in Proposition 7. The only change is that we have to check that

$$\lim_{n\to\infty} b_n \int_{\{|u|\leq n^\delta b_n^{-1}\}} \mathbb{E}\left[e^{-|u|^\beta V_n(A_1+iA_2 sgn(u))}\mathbf{1}_{\Omega_n}\right](\hat{h}_\theta(u)-\hat{h}_\theta(0))\,du = 0\,,$$

which is obviously true since $V_n \geq n^{1-\gamma(1-\beta)_+}$ and since $2\gamma(1-\beta)_+ < 2\delta\beta < 1$, using the fact that $\hat{h}_\theta$ is a Lipschitz function.

Now, since $\hat{h}_\theta$ is bounded, the part corresponding to $n^\delta b_n^{-1} \leq |u| \leq \varepsilon_0 n^{-\gamma}$ is treated as in the proof of Proposition 8 (since it only uses the behavior of φ_ξ around 0, which is the same).

Finally, it remains to prove that

$$\lim_{n\to\infty} b_n \int_{\{|u|\geq\varepsilon_0 n^{-\gamma}\}} e^{-iub_n x}\,\mathbb{E}\left[\prod_x \varphi_\xi(uN_n(x))\mathbf{1}_{\Omega_n}\right]\hat{h}_\theta(u)\,du = 0\,. \tag{24}$$

We note that, if $|u| \geq \varepsilon_0 n^{-\gamma}$ and $x \in \mathbb{Z}^d$, we have

$$\begin{aligned} |\varphi_\xi(uN_n(x))| &\leq \exp(-\sigma|u|^\beta N_n^\beta(x))\,\mathbf{1}_{\{|uN_n(x)|\leq\varepsilon_0\}} + \rho\,\mathbf{1}_{\{|uN_n(x)|\geq\varepsilon_0\}} \\ &\leq \exp(-\sigma\varepsilon_0^\beta n^{-\gamma\beta}N_n^\beta(x))\,\mathbf{1}_{\{|uN_n(x)|\leq\varepsilon_0\}} + \rho\,\mathbf{1}_{\{|uN_n(x)|\geq\varepsilon_0\}}\,. \end{aligned}$$

For n large enough, $\rho \leq \exp(-\sigma\varepsilon_0^\beta n^{-\gamma\beta})$. Therefore, if n is large enough, then for all x and u such that $N_n(x) \geq 1$ and $|u| \geq \varepsilon_0 n^{-\gamma}$, we have

$$|\varphi_\xi(uN_n(x))| \leq \exp(-\sigma\varepsilon_0^\beta n^{-\gamma\beta})\,.$$

Hence,

$$\left|\mathbb{E}\left[\prod_x \varphi_\xi(uN_n(x))\,\mathbf{1}_{\Omega_n}\right]\right| \le \mathbb{E}\left[\exp(-\sigma\varepsilon_0^\beta n^{-\gamma\beta}R_n)\mathbf{1}_{\Omega_n}\right] \le \exp(-\sigma\varepsilon_0^\beta n^{1-\gamma(1+\beta)})\,.$$

Therefore, since $\gamma(1+\beta) < 1$, we have

$$\lim_{n\to\infty} b_n \int_{\{|u|\ge\varepsilon_0 n^{-\gamma}\}} e^{-iub_n x}\,\mathbb{E}\left[\prod_x \varphi_\xi(uN_n(x))\,\mathbf{1}_{\Omega_n}\right]\hat{h}_\theta(u)\,du = 0\,.$$

This concludes the proof of Theorem 3. □

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References

[1] Billingsley, P. *Convergence of probability measures.* Wiley series in probability and mathematical statistics. New York Wiley, 1968.

[2] Bolthausen, E. *A central limit theorem for two-dimensional random walks in random sceneries.* Ann. Probab. **17** (1989), no. 1, 108–115.

[3] Borodin, A. N. *A limit theorem for sums of independent random variables defined on a recurrent random walk.* (Russian) Dokl. Akad. Nauk SSSR **246** (1979), no. 4, 786–787.

[4] Borodin, A. N. *Limit theorems for sums of independent random variables defined on a transient random walk.* Investigations in the theory of probability distributions, IV. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **85** (1979), 17–29, 237, 244.

[5] Castell, F.; Guillotin-Plantard N.; Pène F.; Schapira Br. *A local limit theorem for random walks in random scenery and on randomly oriented lattices* To appear in Ann. Probab..

[6] Cerný, J. *Moments and distribution of the local time of a two-dimensional random walk.* Stochastic Process. Appl. 117 (2007), no. 2, 262–270.

[7] Chen, X. *Moderate and small deviations for the ranges of one-dimensional random walks.* J. Theor. Probab. **19** (2006), 721–739.

[8] Deligiannidis, G.; Utev, S. *An asymptotic variance of the self-intersections of random walks* Arxiv.

[9] Dvoretzky A. and Erdös. P., *Some problems on random walk in space.* Proc. Berkeley Sympos. Math. Statist. Probab. (1951) 353–367.

[10] Durrett, R. *Probability: theory and examples.* Wadsworth & Brooks/Cole. Statistics and Probability Series, Belmont, CA, 1991.

[11] Kesten, H.; Spitzer, F. *A limit theorem related to a new class of self-similar processes.* Z. Wahrsch. Verw. Gebiete **50** (1979), no. 1, 5–25.

[12] Le Doussal, P. *Diffusion in layered random flows, polymers, electrons in random potentials, and spin depolarization in random fields.* J. Statist. Phys. **69** (1992), no. 5-6, 917–954.

[13] Le Gall, J.F.; Rosen, J. *The range of stable random walks.* Ann. Probab. **19** (1991), 650–705.

[14] Matheron, G.; de Marsily G. *Is transport in porous media always diffusive? A counterxample.* Water Resources Res. **16** (1980), 901–907.

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